

SOME PROPERTIES OF PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. Two properties of plurisubharmonic functions are proven. The first result is a Skoda type integrability theorem with respect to a Monge-Ampère mass with Hölder continuous potential. The second one says that locally, a p.s.h. function is k -Lipschitz outside a set of Lebesgue measure smaller than c/k^2 .

1. INTRODUCTION AND MAIN RESULTS

Let Ω be an open subset of \mathbb{C}^n . Recall that a function $\varphi : \Omega \rightarrow [-\infty, \infty)$ is plurisubharmonic¹ (p.s.h. for short) if φ is upper semicontinuous and if for every complex line $L \subset \mathbb{C}^n$ the function $\varphi|_{\Omega \cap L}$ is subharmonic in $\Omega \cap L$.

Basic examples are given by $\varphi := \log |h|$ with $h : \Omega \rightarrow \mathbb{C}$ holomorphic in Ω , in particular $\varphi(z) = \log |z|$ with $\Omega = \mathbb{C}^n$. Plurisubharmonicity is preserved on taking the maxima of a finite number of p.s.h. functions and on taking the pointwise limit of a decreasing sequence of p.s.h. functions.

The *Lelong number* of a p.s.h. function $\varphi : \Omega \rightarrow [-\infty, \infty)$, $\Omega \subset \mathbb{C}^n$, at $a \in \Omega$ is defined as

$$\nu(\varphi; a) := \liminf_{z \rightarrow a, z \neq a} \frac{\varphi(z)}{\log |z - a|}$$

and it somewhat measures the singularity of φ at a . This number can be characterized as $\nu(\varphi; a) = \sup\{\gamma : \varphi(z) \leq \gamma \log |z - a| + O(1) \text{ as } z \rightarrow a\}$ and is one of the most basic quantities associated to the singularity of φ at a . The function $z \mapsto \nu(\varphi; z)$ is upper semicontinuous with respect to the usual topology. It is a deep theorem of Y-T. Siu that this function is also upper semicontinuous with respect to the Zariski topology, i.e. for every $c > 0$ the set $\{a \in \Omega; \nu(\varphi, a) \geq c\}$ is a closed analytic subvariety of Ω . For further properties and equivalent definitions of the Lelong number the reader may consult [Dem] and [Hör07].

A classical theorem of Skoda [Sko72] states that if $\nu(\varphi; a) < 2$ then $e^{-\varphi}$ is integrable in a neighborhood of a with respect to the Lebesgue measure. This result is basic, for instance, in the study of multiplier ideal sheaves associated to a p.s.h. function (see [Laz04]).

Our first result is a generalization of Skoda's theorem where the Lebesgue measure is replaced by a general Monge-Ampère mass with a Hölder continuous local potential (see section 2.1 for the definitions). This class of measures appears naturally in the study of holomorphic dynamical systems (see for instance [Sib99]).

¹We adopt here the standard definition given in [Dem] and [Hör07] and don't exclude functions that are identically $-\infty$ in some connected component of Ω .

Theorem 1.1. *Let $0 < \alpha \leq 1$, let $u : \Omega \rightarrow (-\infty, \infty)$ be an α -Hölder continuous p.s.h. function in the domain $\Omega \subset \mathbb{C}^n$ and let $z \in \Omega$. If φ is a p.s.h. function in Ω and if $\nu(\varphi; z) < \frac{2\alpha}{\alpha+n(2-\alpha)}$, then there is a neighborhood $K \subset \Omega$ of z such that the integral*

$$\int_K e^{-\varphi} (dd^c u)^n$$

is finite. In other words, $e^{-\varphi}$ is locally integrable in $U := \{\xi \in \Omega; \nu(\varphi; \xi) < \frac{2\alpha}{\alpha+n(2-\alpha)}\}$ with respect to the positive measure $(dd^c u)^n$.

Our second result is independent of the first and it applies to a wider class of functions than the class of p.s.h. functions. In what follows, a function $F : \Omega \rightarrow [-\infty, \infty)$ defined on an open subset Ω of \mathbb{C}^n is said to be *separately subharmonic* if for every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ and every $j = 1, \dots, n$ the partial function $z \rightarrow F(\xi_1, \dots, \xi_{j-1}, z, \xi_{j+1}, \dots, \xi_n)$ is subharmonic in its domain of definition (i.e. $U_\xi := \{z \in \mathbb{C}; (\xi_1, \dots, \xi_{j-1}, z, \xi_{j+1}, \dots, \xi_n) \in \Omega\}$).

Roughly speaking, the result says that given $\mathbb{K} \subset \subset \Omega$ and $\varepsilon > 0$, there is a set $L \subset \mathbb{K}$ whose Lebesgue measure is $\leq \varepsilon$ such that F is k -Lipschitz in $\mathbb{K} \setminus L$ with $k \sim \varepsilon^{-2}$. A precise statement is as follows.

Theorem 1.2. *Let F be a negative separately subharmonic function on a connected open subset Ω of \mathbb{C}^n . Let ω and ω' be two non-empty open and relatively compact subsets of Ω . Then for every real number $k > 0$, there is a compact set $L \subset \omega$ such that*

(i) $F|_L$ is finite and k -Lipschitz

(ii) $|\omega \setminus L| \leq \frac{C}{k^2} |F(\xi)|^2$ for every $\xi \in \omega'$,

where C is a positive constant depending only on Ω , ω et ω' .

Acknowledgements. Both authors would like to warmly thank T-C. Dinh whose questions are at the origin of the main results of this paper. The second author was supported by a grant from Région Île-de-France.

2. PRELIMINARY MATERIAL

2.1. Closed positive currents and Monge-Ampère measures. A k -current on a complex manifold X of dimension n is a continuous linear form on the space of compactly supported differential forms of degree $(2n - k)$. Such objects generalize k -forms with coefficients in L^1_{loc} and submanifolds of real codimension k . For the basic theory of currents in complex manifolds see [Dem].

The existence of a complex structure implies, by duality, that every k -current decomposes as a sum of (p, q) -currents with $p + q = k$. Real currents of type (p, p) carry a notion of positivity (see [Dem], [Lel98]). Examples of positive currents include Kähler forms and currents of integration along complex submanifolds of X .

The operators ∂ , $\bar{\partial}$, d and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ extend to currents by duality and the dd^c -Poincaré Lemma states that every positive closed $(1, 1)$ -current T can be written locally as $T = dd^c u$ where u is a p.s.h. function, called the local potential of T .

If T is a closed positive current and u is a locally bounded p.s.h. function the current uT is well-defined and the product (or intersection) current $dd^c u \wedge T$ can be defined by the formula

$$dd^c u \wedge T \stackrel{\text{def}}{=} dd^c(uT).$$

One may then define the product $S \wedge T$ when S is a closed positive $(1, 1)$ -current with bounded local potential: just write $S = dd^c u$ locally and use the above formula. However, the wedge product of general currents may not be well defined.

By induction, the product $dd^c u_1 \wedge \dots \wedge dd^c u_p$ is well-defined when u_1, \dots, u_p are locally bounded p.s.h. functions. In particular, if u is a locally bounded p.s.h. function then the current $(dd^c u)^n$ is well-defined. It is a positive measure called *Monge-Ampère measure* associated to u . See [Kli91] or the original paper [BT82] for some basic properties of the Monge-Ampère operator.

Locally moderate currents. Recall that the set of all p.s.h. functions in X is closed in the space $L_{loc}^1(\Omega)$ and that every family of p.s.h. functions that is bounded in $L_{loc}^1(\Omega)$ is relatively compact in $L_{loc}^1(\Omega)$ (see Theorem 3.2.12 in [Hör07]). For the sake of simplicity, such a family is called a compact family.

The notion of locally moderate currents and measures was introduced by Dinh-Sibony (see [DNS10] for more details).

Definition 2.1. A measure μ on a complex manifold X is called *locally moderate* if for any open set $U \subset X$, any compact set $K \subset U$ and any compact family \mathcal{F} of p.s.h. functions on U there are constants $\beta > 0$ and $C > 0$ such that

$$\int_K e^{-\beta\psi} d\mu \leq C, \text{ for every } \psi \in \mathcal{F}.$$

It follows immediately from the definition that for any \mathcal{F} and μ as above, \mathcal{F} is bounded in $L_{loc}^p(\mu)$ for $1 \leq p < \infty$ and that μ does not charge pluripolar sets.

A positive closed current S of type (p, p) on X is said to be *locally moderate* if the trace measure $\sigma_S = S \wedge \omega^{n-p}$ is locally moderate. Here $n = \dim X$ and ω is the fundamental form of a fixed Hermitian metric on X .

Theorem 2.2. (*Dinh-Nguyen-Sibony* [DNS10]) *If $1 \leq p \leq n$ and u_1, \dots, u_p are Hölder continuous p.s.h. functions on X then the Monge-Ampère current $dd^c u_1 \wedge \dots \wedge dd^c u_p$ is locally moderate.*

The proof in [DNS10] uses the following two lemmas. They will also be used here to prove Theorem 1.1. We denote B_r the ball of radius r centered at the origin of \mathbb{C}^n and fix a fundamental form ω as above.

Lemma 2.3. ([DNS10]) *Let S be a locally moderate closed positive current of type $(n-1, n-1)$ on B_r . If \mathcal{G} is a compact family of p.s.h. functions on B_r then \mathcal{G} is bounded in $L_{loc}^1(\sigma_S)$. Moreover, the mass of the measures $dd^c \varphi \wedge S$, $\varphi \in \mathcal{G}$ are locally bounded in B_r uniformly on φ .*

Proof. (as in [DNS10]) Let K be a compact subset of B_r . After subtracting a fixed constant we may assume that every element of \mathcal{G} is negative on K . Since σ_S is locally

moderate we can choose $\beta, C > 0$ such that $\int_K e^{-\beta\varphi} d\sigma_S \leq C$ for every $\varphi \in \mathcal{G}$. We thus have $\int_K \beta|\varphi| d\sigma_S \leq \int_K e^{-\beta\varphi} d\sigma_S \leq C$ for every $\varphi \in \mathcal{G}$ which proves the first assertion.

For the second assertion let K be a compact subset of B_r and consider a cut-off function χ which is equal to 1 in a neighborhood of K and which is supported on a larger compact $L \subset B_r$. We have, for $\varphi \in \mathcal{G}$

$$\int_K dd^c \varphi \wedge S \leq \int_L \chi dd^c \varphi \wedge S = \int_L dd^c \chi \wedge \varphi S \leq \|\chi\|_{C^2} \int_L |\varphi| d\sigma_S,$$

which is uniformly bounded by the first part of the lemma. \square

Lemma 2.4. ([DNS10]) *Let $r > 0$, S be a locally moderate closed positive current of type $(n-1, n-1)$ on B_{2r} and u be an α -Hölder continuous p.s.h. function on B_r which is smooth on $B_r \setminus B_{r-4\rho}$ for some $0 < \rho < r/4$. Fix a smooth cut-off function χ with compact support in $B_{r-\rho}$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $B_{r-2\rho}$.*

If φ is a p.s.h. function on B_{2r} , then

$$\begin{aligned} \int_{B_r} \chi \varphi dd^c(uS) &= - \int_{B_r \setminus B_{r-3\rho}} dd^c \chi \wedge \varphi u S - \int_{B_r \setminus B_{r-3\rho}} d\chi \wedge \varphi d^c u \wedge S \\ &\quad + \int_{B_r \setminus B_{r-3\rho}} d^c \chi \wedge \varphi du \wedge S + \int_{B_{r-\rho}} \chi u dd^c \varphi \wedge S. \end{aligned}$$

Notice that the smoothness of u in $B_r \setminus B_{r-4\rho}$ makes the second and third integrals meaningful.

Proof. (ref [DNS10]) The case when φ is smooth follows from a direct computation using integration by parts. The general case follows by approximating φ by a decreasing sequence of smooth p.s.h. functions. See [DNS10] for the complete proof. \square

We will also need a volume estimate of the sublevel sets of p.s.h. functions due to M. Kiselman. We include Kiselman's argument here for the the reader's convenience.

Lemma 2.5. ([Kis00]) *Let φ be a p.s.h. function on an open set $\Omega \subset \mathbb{C}^n$ and $K \subset \Omega$ be a compact subset. Then, for every $\gamma < 2/\sup_{z \in K} \nu(\varphi; z)$ there is a constant $C_\gamma = C_\gamma(\varphi, \Omega, K)$ such that*

$$\lambda(K \cap \{\varphi \leq -M\}) \leq C_\gamma e^{-\gamma M}, \quad M \in \mathbb{R},$$

where λ denotes the Lebesgue measure in \mathbb{C}^n .

Proof. Since $e^{\gamma(-M-\varphi)} \geq 1$ on $K \cap \{\varphi \leq -M\}$ we have

$$\mu(K \cap \{\varphi \leq -M\}) \leq \int_K e^{\gamma(-M-\varphi(z))} d\lambda(z) = e^{-\gamma M} \int_K e^{-\gamma\varphi} d\lambda.$$

It suffices then to take $C_\gamma = \int_K e^{-\gamma\varphi} d\lambda$, which is finite by Skoda's Theorem since $\nu(\gamma\varphi; z) < 2$ for every $z \in K$. \square

2.2. Maximal functions and regularity in $W^{1,1}$. If U is open in \mathbb{R}^N and if $f \in L^1_{loc}(U)$, the Lebesgue set of f is

$$\mathcal{L}_f = \left\{ x \in U : \exists a_0 \in \mathbb{R} \text{ such that } \lim_{r \rightarrow 0} \oint_{B(x,r)} |f(t) - a_0| dt = 0 \right\},$$

where the sign \oint_A denotes the average over the set A . When $a_0 = a_0(x)$ exists it is equal to $\tilde{f}(x) := \lim_{r \rightarrow 0} \oint_{B(x,r)} f(x) dx$ and it is well know that (i) \mathcal{L}_f is a Borel set, (ii) $\lambda_N(U \setminus \mathcal{L}_f) = 0$ and (iii) $\tilde{f}(x) = f(x)$ a.e. in U .

For a function $f \in L^1_{loc}(\mathbb{R}^N)$, the Hardy-Littlewood maximal function of f is denoted

$$\mathcal{M}_f(a) := \sup_{r>0} \oint_{B(a,r)} |f(x)| dx, \quad a \in \mathbb{R}^N,$$

More generally, for an open set $U \subset \mathbb{R}^N$ and $f \in L^1_{loc}(\mathbb{R}^N)$ we set

$$\mathcal{M}_f^\rho(a) := \sup_{0 < r < \rho} \oint_{B(a,r)} |f(x)| dx, \quad \rho > 0, a \in U_\rho,$$

where $U_\rho = \{z \in U ; d(z, U^c) > \rho\}$. It is easily checked that \mathcal{M}_f^ρ is Borel measurable in U_ρ .

We may now recall three classical results that will be basic for us in the next section.

Theorem 2.6. (Bojarski [Boj91], Bojarski-Hajlasz [BH93]) *Let $f \in W^{1,1}_{loc}(U)$. If $x \in U_\rho$ is such that $\mathcal{M}_{|\nabla f|}^\rho(x) < +\infty$ then x is a Lebesgue point for f . Furthermore, for every $x, y \in U_\rho$ such that $|x - y| \leq \frac{\rho}{3}$ and $\mathcal{M}_{|\nabla f|}^\rho(x) < \infty$, $\mathcal{M}_{|\nabla f|}^\rho(y) < \infty$ we have*

$$|\tilde{f}(x) - \tilde{f}(y)| \leq C_N |x - y| (\mathcal{M}_{|\nabla f|}^\rho(x) + \mathcal{M}_{|\nabla f|}^\rho(y)),$$

where C_N is a constant depending only on the dimension N .

Theorem 2.7. (Hardy-Littlewood, Wiener. Ref. [AH96]) *Let μ be a finite positive measure on \mathbb{R}^N and let $\mathcal{M}(\mu)(x) = \sup_{B(a,r) \ni x} \frac{\mu(B(a,r))}{|B(a,r)|}$, $x \in \mathbb{R}^N$. Then for every $t > 0$*

$$|\{\mathcal{M}(\mu) > t\}| \leq C \frac{\|\mu\|_1}{t},$$

where C depends only on the dimension N .

Consider now for $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = |x|^{\alpha-N}$ of order α in \mathbb{R}^N . For a finite Radon measure on \mathbb{R}^N the Riesz potential $I_\alpha(\mu)$ is defined by $I_\alpha(\mu)(x) := I_\alpha * \mu(x) = \int |x - y|^{\alpha-N} d\mu(y)$ for $x \in \mathbb{R}^N$.

Theorem 2.8. (Zygmund, cf. [AH96] p. 56) *Let μ be a finite positive Radon measure on \mathbb{R}^N and assume that $0 < \alpha < N$. Then there is a constant A depending only on α and N such that, for every $t > 0$,*

$$|\{I_\alpha(\mu) \geq t\}| \leq \frac{A}{t^{\frac{N}{N-\alpha}}} \|\mu\|_1^{\frac{N}{N-\alpha}}.$$

In fact we need a slightly improved version of this estimate with $I_\alpha(\mu)$ replaced by its maximal function.

Theorem 2.9. *Let μ be a finite positive measure on \mathbb{R}^N , $0 < \alpha < N$, and let $\mathcal{I}_\alpha \mu(x) = \mathcal{M}_{I_\alpha(\mu)}(x)$ be the maximal function of $I_\alpha(\mu)$. Then there is a constant A depending only on α and N such that, for every $t > 0$,*

$$(2.1) \quad |\{\mathcal{I}_\alpha(\mu) \geq t\}| \leq \frac{A}{t^{\frac{N}{N-\alpha}}} \|\mu\|_1^{\frac{N}{N-\alpha}}.$$

Proof. A. We first note an elementary fact. Denote

$$\chi_r = |B(0, r)|^{-1} \mathbf{1}_{B(0, r)}.$$

For $0 < s \leq r$ we have $\chi_s * \chi_r \leq 2^N \chi_{2r}$. Indeed, letting V_N to denote the volume of the unit ball in \mathbb{R}^N , we have $\chi_r \leq V_N^{-1} r^{-N}$ and then $\chi_s * \chi_r \leq V_N^{-1} r^{-N}$ because χ_s is of integral 1. On the other hand, $\chi_s * \chi_r$ vanishes outside $B(0, 2r)$ and $\chi_{2r} = V_N^{-1} (2r)^{-N}$ in $B(0, 2r)$, from where the stated inequality follows.

In particular we have $\mu * \chi_r * \chi_s \leq 2^N \mu * \chi_{2r} \leq 2^N \mathcal{M}(\mu)$ for $0 < s \leq r$. Thus using the commutativity of the convolution, $\mu * \chi_r * \chi_s(x) \leq 2^N \mathcal{M}(\mu)(x)$ for every $r, s > 0$ and on taking the supremum over s we obtain

$$\mathcal{M}(\mu * \chi_r) \leq 2^N \mathcal{M}(\mu).$$

Observe that $I_\alpha(\mu) * \chi_r = I_\alpha * (\mu * \chi_r)$.

B. Next we adapt the argument of Hedberg's proof of Theorem 2.8 (see [AH96] p. 56). Dividing the integral defining $I_\alpha(\mu)(x)$ in two parts we get (exactly as in Hedberg's proof)

$$\begin{aligned} I_\alpha(\mu)(x) &= \int_{y \notin B(x, \delta)} \frac{d\mu(y)}{|x - y|^{N-\alpha}} + \int_{y \in B(x, \delta)} \frac{d\mu(y)}{|x - y|^{N-\alpha}} \\ &\leq \frac{\mu(B(x, \delta))}{\delta^{N-\alpha}} + (N - \alpha) \int_0^\delta \frac{\mu(B(x, t))}{t^{N+1-\alpha}} dt + A \delta^{\alpha-N} \|\mu\|_1 \\ &\leq \delta^\alpha \mathcal{M}(\mu)(x) + A' \delta^\alpha \mathcal{M}(\mu)(x) + A \delta^{\alpha-N} \|\mu\|_1 \\ &\leq C (\delta^\alpha \mathcal{M}(\mu)(x) + \delta^{\alpha-N} \|\mu\|_1). \end{aligned}$$

So we have

$$\begin{aligned} I_\alpha(\mu) * \chi_r(x) &\leq C (\delta^\alpha \mathcal{M}(\mu * \chi_r)(x) + \delta^{\alpha-N} \|\mu * \chi_r\|_1) \\ &\leq C' (\delta^\alpha \mathcal{M}(\mu)(x) + \delta^{\alpha-N} \|\mu\|_1), \end{aligned}$$

where we used the inequality from part A.

Taking the supremum over r we obtain

$$\mathcal{I}_\alpha(x) = \mathcal{M}(I_\alpha(\mu))(x) \leq C' (\delta^\alpha \mathcal{M}(\mu)(x) + \delta^{\alpha-N} \|\mu\|_1),$$

and setting the constant to be $\delta := (\|\mu\|_1 / \mathcal{M}(\mu)(x))^{\frac{1}{N}}$ we get

$$\mathcal{I}_\alpha(x) \leq C \|\mu\|_1^{\alpha/N} (\mathcal{M}(\mu)(x))^{1-(\alpha/N)}.$$

Finally, by the Hardy-Littlewood-Wiener Theorem,

$$\begin{aligned}
|\{\mathcal{I}_\alpha > t\}| &\leq |\{C \|\mu\|_1^{\alpha/N} (\mathcal{M}(\mu))^{1-(\alpha/N)} > t\}| \\
&= |\{(\mathcal{M}(\mu))^{1-(\alpha/N)} > C^{-1} t \|\mu\|_1^{-\alpha/N}\}| \\
&= |\{\mathcal{M}(\mu) > C' t^{\frac{N}{N-\alpha}} \|\mu\|_1^{-\alpha/N-\alpha}\}| \\
&\leq \frac{A}{t^{\frac{N}{N-\alpha}}} \|\mu\|_1^{\frac{N}{N-\alpha}}.
\end{aligned}$$

□

3. INTEGRATION OF P.S.H. FUNCTIONS

This section is devoted to the proof of Theorem 1.1 and some related results.

We will need the following simple extension of the second part of Lemma 2.3.

Lemma 3.1. *Let $1 \leq p \leq n$ and let S be a locally moderate closed positive current of type $(n-p-1, n-p-1)$ on B_r . If \mathcal{G} is a compact family of p.s.h. functions on B_r and \mathcal{H} is locally uniformly bounded family of p.s.h. functions on B_r then the mass of the measures $dd^c \varphi \wedge (dd^c u)^p \wedge S$, $\varphi \in \mathcal{G}$, $u \in \mathcal{H}$ are locally bounded in B_r uniformly on φ and u .*

Proof. Fix a compact subset K of B_r and let $L_0 = K, L_1, \dots, L_p$ be compact subsets of B_r such that L_i is contained in the interior of L_{i+1} . Let χ_i , $i = 1, \dots, p$ be smooth cut-off functions such that $0 \leq \chi_i \leq 1$, $\chi_i \equiv 1$ in L_{i-1} and χ_i is supported in L_i . Then, for $\varphi \in \mathcal{G}$ and $u \in \mathcal{H}$, the mass of $dd^c \varphi \wedge (dd^c u)^p \wedge S$ over K is bounded by

$$\begin{aligned}
\int_{L_1} \chi_1 dd^c \varphi \wedge (dd^c u)^p \wedge S &= \int_{L_1} u (dd^c \chi_1) \wedge dd^c \varphi \wedge (dd^c u)^{p-1} \wedge S \\
&\leq \|\chi_1\|_{C^2} \|u\|_{L^\infty(L_1)} \int_{L_1} dd^c \varphi \wedge (dd^c u)^{p-1} \wedge S \wedge \omega \\
&\leq \|\chi_1\|_{C^2} \|u\|_{L^\infty(L_1)} \int_{L_2} \chi_2 dd^c \varphi \wedge (dd^c u)^{p-1} \wedge S \wedge \omega \\
&= \|\chi_1\|_{C^2} \|u\|_{L^\infty(L_1)} \int_{L_2} u dd^c \chi_2 \wedge dd^c \varphi \wedge (dd^c u)^{p-2} \wedge S \wedge \omega \\
&\leq \|\chi_1\|_{C^2} \|\chi_2\|_{C^2} \|u\|_{L^\infty(L_1)} \|u\|_{L^\infty(L_2)} \int_{L_2} dd^c \varphi \wedge (dd^c u)^{p-2} \wedge S \wedge \omega^2 \\
&\leq \dots \\
&\leq \|\chi_1\|_{C^2} \dots \|\chi_p\|_{C^2} \|u\|_{L^\infty(L_1)} \dots \|u\|_{L^\infty(L_p)} \int_{L_p} dd^c \varphi \wedge S \wedge \omega^p,
\end{aligned}$$

where $\omega = dd^c \|z\|^2$ is the standard fundamental form on \mathbb{C}^n . The result now follows from Lemma 2.3 and the fact that $\|u\|_{L^\infty(L_i)}$ is bounded independently of u . □

Proof of Theorem 1.1. There is no loss of generality in assuming that $z = 0$ and since φ is locally bounded from above we may also assume that φ is negative. As before $\omega = dd^c \|z\|^2$.

The proof is inspired by the methods in [DNS10], Theorem 1.1. It will consist of successive applications of integration by parts formulas (Lemma 2.4) together with a regularization procedure.

For $N > 0$ define $\varphi_N = \max\{\varphi, -N\}$ and $\psi_N = \varphi_{N-1} - \varphi_N$. Notice that $0 \leq \psi_N \leq 1$, ψ_N is supported in $\{\varphi < -N + 1\}$ and $\psi_N \equiv 1$ in $\{\varphi < -N\}$.

Observe that

$$(3.1) \quad \begin{aligned} \int e^{-\varphi} (\mathrm{dd}^c u)^n &= \sum_{N=0}^{\infty} \int_{\{-N \leq \varphi < -N+1\}} e^{-\varphi} (\mathrm{dd}^c u)^n \leq \sum_{N=0}^{\infty} e^N \int_{\{-N \leq \varphi < -N+1\}} (\mathrm{dd}^c u)^n \\ &\leq \sum_{N=0}^{\infty} e^N \int \psi_{N-1} (\mathrm{dd}^c u)^n. \end{aligned}$$

From the hypothesis that $\nu(\varphi; 0) < \frac{2\alpha}{\alpha+n(2-\alpha)}$ and from the upper semicontinuity of the function $z \mapsto \nu(\varphi; z)$ there is an $r > 0$ such that $\sup_{z \in B_{2r}} \nu(\varphi, z) \leq \frac{2\alpha}{\alpha+n(2-\alpha)} - \sigma$ for a small constant $\sigma > 0$. From Lemma 2.5 we get that

$$(3.2) \quad \lambda(B_{2r} \cap \{\varphi \leq -N + 1\}) \lesssim e^{-(\frac{\alpha+n(2-\alpha)}{\alpha} + \delta)N} = e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N},$$

where $\delta > 0$ is a small constant (depending on φ). Here and in what follows the sign \lesssim means that the left-hand sign is smaller or equal than a constant times the right-hand side, the constant being independent from N .

Taking a smaller r if necessary we may assume that u is defined on B_{2r} . Subtracting a constant we may assume that $u \leq -1$. Consider the function $v(z) = \max(u(z), A \log \|z\|)$. If we choose $A > 0$ sufficiently small, we see that v coincides with u near the origin and that $v(z) = A \log \|z\|$ near the boundary of B_r . This allows us to assume that $u(z) = A \log \|z\|$ on $B_r \setminus B_{r-4\rho}$ for some fixed $\rho < r/4$. Notice that, in particular, u is smooth on $B_r \setminus B_{r-4\rho}$.

Fix a smooth cut-off function χ with compact support in $B_{r-\rho}$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $B_{r-2\rho}$. Applying Lemma 2.4 to ψ_{N-1} and $(\mathrm{dd}^c u)^{n-1}$ and noticing that $(\mathrm{dd}^c u)^n = \mathrm{dd}^c(u(\mathrm{dd}^c u)^{n-1})$ we get

$$(3.3) \quad \begin{aligned} &\int_{B_r} \chi \psi_{N-1} (\mathrm{dd}^c u)^n = \\ &- \int_{B_r \setminus B_{r-3\rho}} \mathrm{dd}^c \chi \wedge \psi_{N-1} u (\mathrm{dd}^c u)^{n-1} - \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}\chi \wedge \psi_{N-1} \mathrm{d}^c u \wedge (\mathrm{dd}^c u)^{n-1} \\ &+ \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}^c \chi \wedge \psi_{N-1} \mathrm{d}u \wedge (\mathrm{dd}^c u)^{n-1} + \int_{B_{r-\rho}} \chi u \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1}. \end{aligned}$$

Observing that u is smooth in $B_r \setminus B_{r-3\rho}$, that the support of ψ_{N-1} is contained in $\{\varphi \leq -N + 1\}$ and using the volume estimate (3.2) we get that the absolute values of the first three integrals on the right-hand side are $\leq c_1 e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}$, where $c_1 > 0$ does not depend on N .

For $N \geq 1$ set $\varepsilon = \varepsilon(N) = e^{-(\frac{1}{\alpha} + c)N}$, where $0 < c < \frac{\delta}{n(2-\alpha)}$. Using a convolution with a smooth $U(n)$ -invariant approximation of identity one can choose for N large a

regularization u_ε of u defined on $B_{r-\rho}$ in such a way that $\|u - u_\varepsilon\|_\infty \lesssim \varepsilon^\alpha = e^{-(1+c\alpha)N}$ and $\|u_\varepsilon\|_{C^2} := \|u_\varepsilon\|_{C^2(B_{r-\rho})} \lesssim \varepsilon^{\alpha-2}$.

Writing $u = u_\varepsilon + (u - u_\varepsilon)$ the last integral in (3.3) is equal to

$$\int_{B_{r-\rho}} \chi u_\varepsilon \, dd^c \psi_{N-1} \wedge (dd^c u)^{n-1} + \int_{B_{r-\rho}} \chi (u - u_\varepsilon) \, dd^c \psi_{N-1} \wedge (dd^c u)^{n-1}.$$

Since $\{\varphi_N\}_{N \geq 0}$ is a compact family of p.s.h. functions and since the current $(dd^c u)^{n-1}$ is locally moderate (Theorem 2.2), we see from Lemma 2.3 that the modulus of the second integral above is less than $c_2 \|u - u_\varepsilon\|_\infty \leq c'_2 e^{-(1+c\alpha)N}$ where $c'_2 > 0$ does not depend on N .

To deal with the remaining integral we apply Lemma 2.4 for u_ε instead of u . Noticing that $dd^c(u_\varepsilon \wedge (dd^c u)^{n-1}) = dd^c u_\varepsilon \wedge (dd^c u)^{n-1}$ we get

$$\begin{aligned} & \int_{B_{r-\rho}} \chi u_\varepsilon \, dd^c \psi_{N-1} \wedge (dd^c u)^{n-1} = \\ & \int_{B_r \setminus B_{r-3\rho}} dd^c \chi \wedge \psi_{N-1} u_\varepsilon (dd^c u)^{n-1} + \int_{B_r \setminus B_{r-3\rho}} d\chi \wedge \psi_{N-1} dd^c u_\varepsilon \wedge (dd^c u)^{n-1} \\ & - \int_{B_r \setminus B_{r-3\rho}} d^c \chi \wedge \psi_{N-1} du_\varepsilon \wedge (dd^c u)^{n-1} + \int_{B_r} \chi \psi_{N-1} dd^c u_\varepsilon \wedge (dd^c u)^{n-1}. \end{aligned}$$

Since $u(z) = A \log \|z\|$ on $B_r \setminus B_{r-4\rho}$ the C^2 norm of u_ε on $B_r \setminus B_{r-3\rho}$ does not depend on $\varepsilon = \varepsilon(N)$. Together with the volume estimate (3.2) this implies that the first three integrals in the right-hand side have absolute values less than $c_3 e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}$ where $c_3 > 0$ does not depend on N .

For the last integral we write $dd^c u_\varepsilon \wedge (dd^c u)^{n-1} = dd^c(u dd^c u_\varepsilon \wedge (dd^c u)^{n-2})$ and apply Lemma 2.4 for $S = dd^c u_\varepsilon \wedge (dd^c u)^{n-2}$. This gives us four integrals. Three of them are integrals over $B_r \setminus B_{r-3\rho}$ involving u , u_ε , ψ_{N-1} and its derivatives. As above, the absolute value of each one of them is $\lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}$. The remaining integral is

$$\int_{B_{r-\rho}} \chi u dd^c \psi_{N-1} \wedge dd^c u_\varepsilon \wedge (dd^c u)^{n-2},$$

which we write again as

$$(3.4) \quad \int_{B_{r-\rho}} \chi u_\varepsilon dd^c \psi_{N-1} \wedge dd^c u_\varepsilon \wedge (dd^c u)^{n-2} + \int_{B_{r-\rho}} \chi (u - u_\varepsilon) dd^c \psi_{N-1} \wedge dd^c u_\varepsilon \wedge (dd^c u)^{n-2}.$$

Since u_ε converges to u in L^∞ , Lemma 3.1 implies that the mass of $dd^c \psi_{N-1} \wedge dd^c u_\varepsilon \wedge (dd^c u)^{n-2}$ is bounded independently of N and ε . Therefore, the modulus of the second integral above is less than $c_4 \|u - u_\varepsilon\|_\infty \leq c'_4 e^{-(1+c\alpha)N}$ where $c'_4 > 0$ does not depend on N .

To deal with the the first integral in (3.4) we apply Lemma 2.4, obtaining three integrals over $B_r \setminus B_{r-3\rho}$ whose absolute values are $\lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}$ and the integral

$$\int_{B_{r-\rho}} \chi \psi_{N-1} (dd^c u_\varepsilon)^2 \wedge (dd^c u)^{n-2}.$$

We can repeat the above procedure in order “move” the dd^c ’s from u to u_ε . We get at each step integrals with absolute values $\lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}$ or $\lesssim e^{-(1+c\alpha)N}$ (where the constants involved don’t depend on N) and at the final step we get the integral

$$\int_{B_{r-\rho}} \chi \psi_{N-1} (\text{dd}^c u_\varepsilon)^n,$$

whose absolute value is less than $c_5 \|u_\varepsilon\|_{\mathcal{C}^2}^n \cdot \lambda\{\varphi \leq -N+1\} \leq c'_5 \varepsilon^{n(\alpha-2)} e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N} = c'_5 e^{(cn(2-\alpha)-(1+\delta))N}$, with c'_5 independent from N .

Altogether the above estimates yield

$$\int_{B_{r-\rho}} \chi \psi_{N-1} (\text{dd}^c u)^n \lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N} + e^{-(1+c\alpha)N} + e^{(cn(2-\alpha)-(1+\delta))N}.$$

Inserting these estimates in (3.1) we finally get

$$\begin{aligned} \int_{B_r} e^{-\varphi} (\text{dd}^c u)^n &\lesssim \sum_{N=0}^{\infty} e^N \left[e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N} + e^{-(1+c\alpha)N} + e^{(cn(2-\alpha)-(1+\delta))N} \right] \\ &= \sum_{N=0}^{\infty} \left[e^{-\delta N - \frac{n(2-\alpha)}{\alpha}N} + e^{-c\alpha N} + e^{(cn(2-\alpha)-\delta)N} \right]. \end{aligned}$$

By the choice of c all the factors of N in the exponentials above are negative, so the series converges and hence the integral $\int e^{-\varphi} (\text{dd}^c u)^n$ is finite. \square

From Theorem 1.1 and a computation analogous to the one made in the proof of Lemma 2.5 follows an estimate of the measure of the sub-level sets of p.s.h. functions with respect to Monge-Ampère masses with Hölder continuous potential.

Corollary 3.2. *Let φ be a p.s.h. function on an open set $\Omega \subset \mathbb{C}^n$ and $\mu = (\text{dd}^c u)^n$ a Monge-Ampère mass on Ω with u an α -Hölder continuous p.s.h. function. If $K \subset \Omega$ is a compact subset then for every $\gamma < \frac{2\alpha}{\alpha+n(2-\alpha)} \frac{1}{\sup_{z \in K} \nu(\varphi; z)}$ there is a constant $C_\gamma = C_\gamma(\varphi, \Omega, K)$ such that*

$$\mu(K \cap \{\varphi \leq -M\}) \leq C_\gamma e^{-\gamma M}, \quad M \in \mathbb{R}.$$

Another theorem of Skoda concerns the non-integrability of a p.s.h. function with large Lelong number: if $\nu(\varphi; 0) > 2n$ then $e^{-\varphi}$ is not integrable in any neighborhood of the origin with respect to Lebesgue measure (see [Hör07] Lemma 4.3.1). One cannot hope for a similar result with respect to every Monge-Ampère measure with Hölder continuous potential, because the measure $\mu = (\text{dd}^c u)^n$ can be arbitrarily small near 0 (and even zero), making the integral $\int e^{-\varphi} d\mu$ finite. We may note however the following fact.

Proposition 3.3. *Fix $0 < \alpha \leq 1$. There exists a Monge-Ampère mass $\mu = (\text{dd}^c u)^n$ where u is an α -Hölderian p.s.h. function such that for every p.s.h. function φ defined near 0 with $\nu(\varphi; 0) > n\alpha$ we have $\int_K e^{-\varphi} d\mu = +\infty$ for every neighborhood K of the origin.*

Proof. Let $\gamma = \nu(\varphi; 0) > n\alpha$. Since $\varphi(z) \leq \gamma \log \|z\| + O(1)$ near 0 (see Introduction) we have that $e^{-\varphi(z)} \geq C \frac{1}{\|z\|^\gamma} \geq C \frac{1}{\|z\|^{n\alpha}}$. If we take $u(z) = \|z\|^\alpha$ then a direct computation shows that $(dd^c u)^n = C^{st} \|z\|^{n(\alpha-2)} \cdot \lambda$ in the sense of currents. We thus have

$$\int e^{-\varphi} (dd^c u)^n \geq C^{st} \int \frac{1}{\|z\|^{n\alpha}} \frac{1}{\|z\|^{n(2-\alpha)}} d\lambda = C^{st} \int \frac{1}{\|z\|^{2n}} d\lambda,$$

and the last integral diverges in any neighborhood of the origin. \square

Remark 3.4. For $n = 1$ the condition on the Lelong number of φ on the hypothesis of Theorem 1.1 is $\nu(\varphi; z) < \alpha$. This bound is sharp as Proposition 3.3 shows.

Remark 3.5. For $n \geq 2$ the condition $\nu(\varphi; z) < \frac{2\alpha}{\alpha+n(2-\alpha)}$ in Theorem 1.1 is probably no longer optimal, as the example below suggest.

Let $0 < \alpha \leq 1$ and $c > 0$. Consider the potential $u(z) = |z_1|^\alpha + \dots + |z_n|^\alpha$ and the p.s.h. function $\varphi(z) = c \log |z_1|$ defined on \mathbb{C}^n . We have then that $(dd^c u)^n = n! \left(\frac{\alpha}{2}\right)^{2n} |z_1|^{\alpha-2} \dots |z_n|^{\alpha-2} \cdot \lambda$ as measures on \mathbb{C}^n . Notice that this expression makes sense, since $\alpha > 0$ implies $|z_1|^{\alpha-2} \dots |z_n|^{\alpha-2} \in L^1_{loc}(\mathbb{C}^n)$.

We thus have

$$\begin{aligned} \int_{B_1} e^{-\varphi} (dd^c u)^n &= n! \left(\frac{\alpha}{2}\right)^{2n} \int_{B_1} \frac{1}{|z_1|^c} |z_1|^{\alpha-2} \dots |z_n|^{\alpha-2} d\lambda \\ &= n! \left(\frac{\alpha}{2}\right)^{2n} \int_{B_1} \frac{1}{|z_1|^{c-\alpha+2}} |z_2|^{\alpha-2} \dots |z_n|^{\alpha-2} d\lambda, \end{aligned}$$

which is finite if and only if $\nu(\varphi; 0) = c < \alpha$.

4. REGULARITY OF P.S.H. FUNCTIONS

This section is devoted to the proof Theorem 1.2. Since our approach is rather in the spirit of classical potential theory it will be convenient to deal with (*positive*) *superharmonic functions*, instead of (negative) subharmonic functions. We refer the reader to [AG01] [Bre69], [Hel09] for the basic potential theoretic notions used in what follows.

Let us start with the one complex dimensional case, that is, that of superharmonic functions on an open subset of \mathbb{C} .

Theorem 4.1. *Let F be a positive superharmonic function on a connected open subset Ω of \mathbb{C} and let \mathbb{K} be a relatively compact subset of Ω . Let ω be a fixed open neighborhood of \mathbb{K} which is relatively compact in Ω and set $I_{F,\omega} = \min\{F(x) : x \in \omega\}$.*

Then for every real number $k > 0$ there is a compact set $L \subset \mathbb{K}$ such that

(i) $F|_L$ is finite and k -Lipschitz

(ii) $|\mathbb{K} \setminus L| \leq \frac{C}{k^2} (I_{F,\omega})^2$,

where C is a positive constant depending only on Ω , ω and \mathbb{K} .

Remark 4.2. Since F is superharmonic, $F(a) = \lim_{r \rightarrow 0} \oint_{B(a,r)} F(x) dx$ for $a \in \Omega$ and so $\tilde{F} = F$ on the Lebesgue set \mathcal{L}_F . Since F is l.s.c. it follows that $\mathcal{L}_F = \Omega \cap \{F < +\infty\}$.

Proof. We can assume that Ω is bounded and replacing F by its reduction (or *réduite*) over ω -with respect to Ω - (see [Bre69], [Hel09], [AG01] or the proof of Lemma 4.5) we can assume that F is a potential in Ω , i.e., the greatest harmonic minorant of F in Ω is zero, and that $\mu = -\Delta F$ is a positive measure supported in $\bar{\omega}$. Thus, F is positive and harmonic in $\Omega \setminus \bar{\omega}$ and vanishes on $\partial\Omega$.

Since $F = G(\mu)$ is the Green potential of μ in Ω and $G(z, w) \geq c$ in $\bar{\omega} \times \bar{\omega}$ we have that $\|\mu\|_1 \leq C^{ste} I_{F, \omega}$. Therefore, in order to prove the theorem it suffices to find L such that F is k -Lipschitz on L and $|\mathbb{K} \setminus L| \leq \frac{C}{k^2} \|\mu\|_1^2$.

Write $F = N * \mu + H$ where $N(z) = \frac{1}{2\pi} \log \frac{1}{\|z\|}$ and H is a harmonic function on Ω . Setting $R := \sup\{|z - z'|; z \in \bar{\omega}, z' \in \partial\Omega\}$ and $r := \inf\{|z - z'|; z \in \bar{\omega}, z' \in \partial\Omega\}$ we have over $\partial\Omega$ the inequalities

$$\frac{1}{2\pi} \log \left(\frac{1}{R} \right) \|\mu\|_1 \leq N * \mu \leq \frac{1}{2\pi} \log \left(\frac{1}{r} \right) \|\mu\|_1,$$

which implies that H is bounded by two fixed multiples of $\|\mu\|_1$ in Ω . By the Harnack property (or the Poisson formula), we conclude that for ω' a relatively compact open subset Ω we have $\|\nabla H\|_{L^\infty(\omega')} \leq C \|\mu\|_1$ for a constant $C > 0$ depending only on N , Ω et ω' . Taking ω' to be a connected neighborhood of $\bar{\omega}$ we see that H is $c\|\mu\|_1$ -Lipschitz over ω where $c = c(N, \omega', \omega)$. It suffices then to prove (i) and (ii) for $s := N * \mu$ instead of F .

Notice that $s \in W_{loc}^{1,p}(\Omega)$ for $1 \leq p < 2$ and $\frac{\partial s}{\partial x_j} = \frac{\partial N}{\partial x_j} * \mu = -\frac{1}{2\pi} \frac{x_j}{|x|^2} * \mu$. From the fact that $|\frac{x_j}{|x|^2}| \leq \frac{1}{|x|}$ and Theorem 2.9 it follows that the maximal function of $|\nabla s|$ satisfies the weak type L^2 inequalities,

$$(4.1) \quad |\{\mathcal{M}_{|\nabla s|} \geq t\}| \leq \frac{C}{t^2} \|\mu\|_1^2, \quad t > 0.$$

Given $k > 0$, let $A = \{\mathcal{M}_{|\nabla s|} \leq \frac{k}{2C_2}\}$, where C_2 is the constant appearing in Theorem 2.6. From this theorem and Remark 4.2 we get

$$|s(z) - s(w)| \leq k|z - w| \text{ for every } z, w \in A,$$

and by (4.1) $|\mathbb{C} \setminus A| \leq \frac{C}{k^2} \|\mu\|_1^2$. Since λ_2 is inner regular, the proof is complete. \square

We now proceed to the proof of Theorem 1.2, stated in terms of separately superharmonic functions.

Theorem 4.3. *Let F be a positive separately superharmonic function on a connected open subset Ω of \mathbb{C}^n . Let ω and ω' be two non-empty open and relatively compact subsets of Ω . Then for every real number $k > 0$, there is a compact set $L \subset \omega$ such that*

(i) $F|_L$ is finite and k -Lipschitz

(ii) $|\omega \setminus L| \leq \frac{C}{k^2} |F(\xi)|^2$ for every $\xi \in \omega'$,

where C is a positive constant depending only on Ω , ω et ω' .

Remark 4.4. By a theorem of Avaniissian [Ava61], we know that F is lower semicontinuous and superharmonic in Ω .

Proof. We may assume $F \not\equiv +\infty$. The result being local (cf. Lemmas 4.5 and 4.7 below) we can assume $\omega = \omega' = D(0, 1)^n$ and $\Omega = D(0, 4)^n$.

A. For a fixed $\xi := (\xi_1, \xi_2, \dots, \xi_n) \in D(0, 4)^n$ let F_ξ denote the partial function $D(0, 4) \ni z \mapsto F(z, \xi_2, \dots, \xi_n)$, which is superharmonic (possibly $\equiv +\infty$). Denoting $\mathcal{N}_1 := \{\xi \in D(0, 4)^n; F_\xi \equiv +\infty\}$, the partial gradient $\nabla_1 F(\cdot, \xi_2, \dots, \xi_n)$ is well-defined and its absolute value belongs to $L^1_{loc}(D(0, 4))$ for $\xi \notin \mathcal{N}_1$. For $\xi \in \mathcal{N}_1$ we set the convention that $|\nabla_1 F(\cdot, \xi_2, \dots, \xi_n)| \equiv +\infty$.

We may define then the partial (local) maximal function $\mathcal{M}^2_{|\nabla_1 F|}$ over $D(0, 1)^n$. It is the positive *everywhere defined* Borel function given by

$$\mathcal{M}^2_{|\nabla_1 F|}(z) := \sup_{0 < r \leq 2} \oint_{D(z_1, r)} |\nabla_1 F(\xi_1, z_2, \dots, z_n)| d\xi_1.$$

We can define analogously the exceptional sets \mathcal{N}_j , the partial gradients $\nabla_j F$ and the respective maximal functions $\mathcal{M}^2_{|\nabla_j F|}$ for $j = 2, \dots, n$.

B. Let us denote $D = D(0, 1)$. Fix $k > 0$ and define, for $j = 1, \dots, n$, the sets

$$A^{(j)} = \{z \in D^n; \mathcal{M}^2_{|\nabla_j F|}(z) \leq c_0 k\},$$

where $c_0 > 0$ is a small constant, chosen independently of F and n in such a way that F is k -Lipschitz over every $A^{(j)} \cap \{z : z_k = z_k^0, \text{ for every } k \neq j\}$ (see Theorem 2.6). By Theorem 4.1 we have that

$$\lambda_2((D^n \setminus A^{(j)}) \cap \{z : z_\ell = \xi_\ell \text{ pour } \ell \neq j\}) \leq \frac{c_1}{k^2} F(\xi)^2$$

for every $\xi \in D^n$, where c_1 is a constant (notice that the inequality remains true if $\xi \in \mathcal{N}_j$). Integrating in ξ_ℓ , $\ell \neq j$, using Fubini's Theorem and Lemma 4.5 we get

$$\lambda_{2n}((D^n \setminus A^{(j)}) \leq \frac{c_2}{k^2} F(\zeta)^2$$

for every $\zeta \in D^n$. Setting $B_1 := \bigcap_{j=1, \dots, n} A^{(j)}$ we get a Borel subset of D^n such that (i) $\lambda_{2n}(D^n \setminus B_1) \leq \frac{c_2 n}{k^2} F(\zeta)^2$ for every $\zeta \in D^n$ and (ii) $F|_{B_1}$ is k -Lipschitz in each variable. Notice that the constant c_2 depends only on n .

Let $\alpha \in (0, 1)$ a constant depending only on n which will fix later. Throwing away the points of B_1 whose density relative to D with respect to the second variable is $\leq \alpha$ we get by Lemma 4.10 a new Borel set $B_2 \subset B_1$ such that $\lambda_{2n}(D^n \setminus B_2) \leq \frac{c_3}{k^2} F(\xi)^2$ with a new constant $c_3 = c_3(n)$. Repeating this procedure with respect to the other variables we get $B_1 \supset B_2 \supset \dots \supset B_n = B$ such that $\lambda_{2n}(D^n \setminus B_n) \leq \frac{c_4}{k^2} F(\xi)^2$ with the property that all points of B_p have density $\geq \alpha$ relatively to D with respect to the first p variables.

C. Let us show now that with this construction, for every pair of points $u, v \in B$ such that $|u_1 - v_1| \geq |u_2 - v_2| \geq \dots \geq |u_n - v_n|$, we have $|F(u) - F(v)| \leq nk |u - v|$.

To this end we show by induction on p that if $u, v \in B_p$, and $u_j = v_j$ for $j > p$ we have $|F(u) - F(v)| \leq 2pk |u - v|$. For the sake of simplicity we treat the step from $p = n - 1$ to $p = n$, the proof for the general step being similar.

Denote $u = (u', u'') \in \mathbb{C}^{n-1} \times \mathbb{C}$, $v = (v', v'') \in \mathbb{C}^{n-1} \times \mathbb{C}$ and $\rho = |v'' - u''| \leq |u' - v'|$. The sections $T_{u'}$ et $T_{v'}$ of B_{n-1} in the fibers $\{u'\} \times D$ and $\{v'\} \times D$ have u'' and v'' as points with density $\geq \alpha$. By Lemma 4.8, if α is chosen to be close enough to 1, there is a $w \in D$ such that $(u', w) \in T_{u'}$, $(v', w) \in T_{v'}$, $|w - u''| \leq \rho$ and $|w - v''| \leq \rho$. Therefore

$$|F(u) - F(v)| \leq |F(u', u'') - F(u', w)| + |F(u', w) - F(v', w)| + |F(v', w) - F(v', v'')|.$$

We know that $F|_{B_1}$ is k -Lipschitz in the last variable and by the induction hypothesis, $F(\cdot, w)$ is $2(n-1)k$ -Lipschitz on $B_{n-1} \cap \{(z', z'') : z'' = w\}$. Therefore

$$|F(u) - F(v)| \leq 2k\rho + 2(n-1)k|u' - v'| \leq 2nk|u' - v'| \leq 2nk|u - v|.$$

D. We have thus shown that given F separately superharmonic on $D(0, 4)^n$ and $k \geq 1$, there is a Borel subset $B \subset D^n$ such that (i) $\lambda_{2n}(D^n \setminus B) \leq \frac{c}{k^2} F(\xi)^2$, for every $\xi \in D^n$, $c = c(n)$ and (ii) $|F(u) - F(v)| \leq k|u - v|$ for every $u, v \in B$ satisfying $|u_1 - v_1| \geq \dots \geq |u_n - v_n|$. By choosing a permutation σ of $\{1, \dots, n\}$ the Lipschitz condition still holds if u, v in (ii) satisfy $|u_{\sigma(1)} - v_{\sigma(1)}| \geq \dots \geq |u_{\sigma(n)} - v_{\sigma(n)}|$. Replacing B by the intersection of the $n!$ sets obtained in this way we get a new set $A \subset D^n$ such that (i) $\lambda_{2n}(D^n \setminus A) \leq \frac{c'}{k^2} F(\xi)^2$, for $\xi \in D^n$, $c' = c'(n)$ and (ii) $F|_A$ is k -Lipschitz in B . Finally, the existence of the compact L and the constant C follows from the inner regularity of λ_{2n} . □

4.1. Auxiliary lemmas. This first lemma tells us that for a positive separately superharmonic function f on a domain Ω of \mathbb{C}^n , the quantities $\int_{\omega} |f(z)|^2 d\lambda_{2n}(z)$ and $\inf\{f(z)^2 : z \in \omega\}$ are in some sense equivalent and independent of the chosen open subset $\omega \subset \subset \Omega$.

Lemma 4.5. *Let Ω be an open connected subset of \mathbb{C}^n and let ω, ω' two non-empty open sets, relatively compact in Ω . Then for every positive separately superharmonic function $f : \Omega \rightarrow \overline{\mathbb{R}}_+$ and every $z \in \omega'$ we have*

$$C^{-1} \inf\{f(\xi)^2; \xi \in \omega'\} \leq \int_{\omega} |f(x)|^2 d\lambda_{2n}(x) \leq C f(z)^2$$

where $C = C(\Omega; \omega, \omega')$ is a finite positive constant depending only on Ω, ω and ω' .

Proof. A. Case $n = 1$. We may assume that Ω is bounded. For the right side inequality we can restrict ourselves, by replacing f by its reduction -or *réduite*- (with respect to Ω) over ω

$$R_F^{\omega} = \inf\{u : u \text{ is positive superharmonic in } \Omega \text{ and } u \geq F \text{ in } \omega\}$$

to the case where $f = G\mu$ is the Green potential, in Ω , of a finite measure supported in $\overline{\omega}$, that is $f(z) = \int G(z, z') d\mu(z')$.

Since $G(z, z') \leq C + \frac{1}{2\pi} \log(\frac{1}{|z - z'|})$ on $\omega \times \omega$ for a constant $C = C(\Omega, \omega)$ we get $\int_{\omega} |f(\xi)|^2 d\lambda_2(\xi) \leq C \|\mu\|_1^2$ and since $G(z, \xi) \geq c = c(\Omega, \omega, \omega')$ for $(z, \xi) \in \omega \times \omega'$, the right side inequality follows.

For the other inequality, we may replace f by its reduction over ω' and suppose $f = G\mu$ with μ supported in $\overline{\omega'}$. We can even assume, by approximating this reduced function,

that μ is supported on a compact set $L \subset \subset \omega'$ with non-empty interior. By Hölder inequality and balayage definition we have

$$\lambda_2(\omega) \int_{\omega} |f(\xi)|^2 d\lambda_2(\xi) \geq \left[\int_{\omega} f(\xi) d\lambda_2(\xi) \right]^2 = \left[\int_{\omega} f(\xi) d\nu_L(\xi) \right]^2,$$

where ν_L is the balayage (or swept out measure) of $1_{\omega} \lambda_2$ over L , relatively to Ω (by definition $R_{G(1_{\omega} \lambda_2)}^L = G(\nu_L)$). As $|\nu_L| \neq 0$ we get the desired inequality: $\int_{\omega} |f(\xi)|^2 d\lambda_2(\xi) \geq c(\omega', \omega, \Omega) \inf_{\xi \in \omega'} f(\xi)^2$.

B. Case $n \geq 2$. We can easily reduce the problem to the case where ω and ω' are both open polydiscs $\prod_{j=1}^n D_j$ with compact closure in Ω .

Notice that when $\omega = \omega'$, the result follows with no greater difficulty: the first inequality is trivial and the second one follows from part A and Fubini's Theorem. In the case ω and ω' are distinct polydiscs it suffices to show that $\inf_{\omega} f \geq c(\Omega, \omega, \omega') \inf_{\omega'} f$ which is the content of Lemma 4.6 below. \square

The following elementary lemma can be seen as an extension of the Harnack inequalities to superharmonic functions.

Lemma 4.6. *Let Ω be a connected open subset of \mathbb{R}^N . Then for every $\delta > 0$, $\eta > 0$, there is a $c = c(\Omega, \delta, \eta) > 0$ such that for every positive superharmonic function f in Ω and every $z, z' \in \Omega(\delta) := \{m \in \Omega; d(m, \partial\Omega) \geq \delta, |m| \leq \delta^{-1}\}$ we have*

$$f(z) \geq c \inf\{f(\xi); |\xi - z'| \leq \eta\}$$

Remark. After perhaps diminishing η , it suffices to treat the case where $\eta \leq \delta/2$. On replacing then δ by 2η , we may assume that $\eta = \delta/2$.

Proof. If $a, b \in \Omega(\delta)$, $b \in \Omega$ are such that $d(a, b) \leq \delta/4$ we have, for $a' \in B(a, \delta/4)$,

$$f(a') \geq \frac{4^N}{3^N v_N \delta^N} \int_{B(a', 3\delta/4)} f(z) d\lambda_N(z) \geq \frac{1}{3^N} \inf\{f(b'); |b - b'| \leq \delta/4\}.$$

So if we denote $m_{f,\delta}(z) := \inf\{f(\xi); \xi \in \Omega, |\xi - z| \leq \delta/4\}$, have

$$m_{f,\delta}(a) \geq \frac{1}{3^N} m_{f,\delta}(b).$$

b) Let $\delta > 0$. Fix a connected compact set $\mathbb{K} \subset \Omega$ containing $\Omega(\delta)$. We can cover \mathbb{K} by a finite number of balls $B(a_j; \delta'/8)$, $1 \leq j \leq \ell$ where $\delta' > 0$ is chosen in such a way that $\mathbb{K} \subset \Omega(\delta')$. Since \mathbb{K} is connected, any two points $m, m' \in \mathbb{K}$ can be joined by a $\frac{\delta'}{4}$ -chain $\{m; a_{i_1}; \dots; a_{i_k}; \dots; a_{i_\nu}; m'\}$ of points of \mathbb{K} (with $\nu \leq \ell$). From the part a) we get $f(m) \geq c \inf\{f(z); |z - m'| \leq \delta'/4\}$. \square

Lemma 4.7. *Let $\overline{D}_n(a_j, r) := \prod_{p=1}^n \overline{D}(a_j^p, r)$, $1 \leq j \leq \ell$, a sequence of closed polydiscs of same radius in \mathbb{C}^n whose union \mathbb{L}_0 is connected and let $F : \mathbb{L}_1 := \bigcup_{1 \leq j \leq \ell} \overline{D}_n(a_j, 2r) \rightarrow \mathbb{R}$ be a real function. Suppose that there exists a constant $c > 0$ such that for every $k > 0$ and every $j = 1, \dots, \ell$ there is a compact subset $\mathbb{K}_{k,j} \subset \overline{D}_n(a_j, 2r)$ such that (i) $F|_{\mathbb{K}_{k,j}}$ is k -Lipschitz and (ii) $\lambda_{2n}(\overline{D}_n(a_j, 2r) \setminus \mathbb{K}_{k,j}) \leq \frac{c}{k^2}$.*

Then there exists $c' > 0$ and for every $k > 0$ a compact subset $\mathbb{K}_k \subset \mathbb{K}$ such that:

(a) $F|_{\mathbb{K}_k}$ is k -lipschitz

(b) $\lambda_{2n}(\mathbb{L}_0 \setminus \mathbb{K}_k) \leq \frac{c'}{k^2}$.

Moreover, we can chose c' as depending only on c and on the sequence $\{D_n(a_j, r)\}_{1 \leq j \leq \ell}$.

Proof. Notice first that it suffices to show that the properties (a) and (b) hold for k bigger than some $k_0 = k_0(n, \ell, c, r)$ since the case of arbitrary k will follow by replacing c' by a bigger constant.

Set $\omega_{k,j} = \overline{D}_n(a_j, 2r) \setminus \mathbb{K}_{k,j}$, $\mathbb{K}_k^{(1)} = \mathbb{L}_1 \setminus \cup_j \omega_{k,j}$, $\mathbb{K}_k^{(0)} = \mathbb{L}_0 \setminus \cup_j \omega_{k,j} = \mathbb{K}_k^{(1)} \cap \mathbb{L}_0$. By assumption, $\lambda_{2n}(\omega_{k,j}) \leq \frac{c}{k^2}$, $\omega_{k,j} \subset \overline{D}_n(a_j, 2r)$, and $|f(m) - f(m')| \leq k|m - m'|$ for $m, m' \in D_n(a_j, 2r) \cap \mathbb{K}_k^{(1)}$, $1 \leq j \leq \ell$.

In particular, if $\overline{D}_n(a_{j_1}, r) \cap \overline{D}_n(a_{j_2}, r) \neq \emptyset$, and k is big enough ($k \geq k_0(n, r, c)$), then $\mathbb{K}_k^{(0)} \cap \overline{D}_n(a_{j_1}, 2r) \cap \overline{D}_n(a_{j_2}, 2r) \neq \emptyset$ and hence $|f(m_1) - f(m_2)| \leq 2nkr$ for $m_1 \in \mathbb{L}_0 \cap \overline{D}_n(a_{j_1}, r)$, $m_2 \in \mathbb{L}_0 \cap \overline{D}_n(a_{j_2}, r)$. From the connectedness of \mathbb{L}_0 we get that $|f(m) - f(m')| \leq 2n\ell kr$ for $m, m' \in \mathbb{K}_k^{(0)}$.

Therefore, if m, m' are points in $\mathbb{K}_k^{(0)}$ that do not belong to the same polydisc $\overline{D}_n(a_j, 2r)$, we have $|m - m'| \geq 2r$ and (for $k \geq k_0$)

$$|f(m) - f(m')| \leq 2n\ell kr = 2n\ell k \frac{r}{|m - m'|} \times |m - m'| \leq c_2 k |m - m'|,$$

where $c_2 = n\ell$.

We see then that (a) and (b) hold for $k \geq c_2 k_0$ if we set $\mathbb{K}_k = \mathbb{K}_{c_2^{-1}k}^{(0)}$ and $c' = (c_2)^2 c\ell$. □

In the following we say that a Borel subset A of the disk $D(0, 1)$ is of density $\geq \alpha$ at z relatively to $D(0, 1)$ if $\lambda_2(A \cap D(z, \rho) \cap D(0, 1)) \geq \alpha \lambda_2(D(z, \rho) \cap D(0, 1))$, for all $\rho > 0$. Notice that this is much stronger than the usual notion of a density at a larger than α .

Lemma 4.8. *Let $z, z' \in D(0, 1)$, $r = |z - z'|$, and $A, A' \subset D(0, 1)$. Then there is a constant α_0 with the following property: if relatively to $D(0, 1)$, A and A' are of density $\geq \alpha_0$ at z, z' respectively, then $A \cap A' \cap D(z, r) \cap D(z', r)$ is non-empty.*

Proof. If $A \cap A' \cap D(z, r) \cap D(z', r) = \emptyset$, then either $\lambda_2(A \cap D(z, r) \cap D(z', r)) \leq \frac{1}{2} \lambda_2(D(z, r) \cap D(z', r) \cap D(0, 1))$ or $\lambda_2(A' \cap D(z, r) \cap D(z', r)) \leq \frac{1}{2} \lambda_2(D(z, r) \cap D(z', r) \cap D(0, 1))$. In the first case $\lambda_2(D(z, r) \cap D(0, 1) \setminus A) \geq \frac{1}{2} \lambda_2(D(z, r) \cap D(z', r) \cap D(0, 1)) \geq c_0 \lambda_2(D(z, r) \cap D(0, 1))$ for some constant c_0 . This means that A is of density $\leq 1 - c_0$ in z relatively to $D(0, 1)$. A similar argument applies in the second case using A', z' instead of A, z . Hence, it suffices to take $\alpha_0 = 1 - c_0$. □

Lemma 4.9. *Let $A \subset \mathbb{R}^N$ be a Borel set contained in a cube (or an open ball) C_0 and let $\alpha \in (0, 1)$. Denote $A_N(\alpha)$ the set of points of A where A is of density $\geq \alpha$ relatively to C_0 . Then $A_N(\alpha)$ is Borel-measurable and*

$$\lambda_N(C_0 \setminus A_N(\alpha)) \leq c(N, \alpha) \lambda_N(C_0 \setminus A)$$

for a finite constant $c(N, \alpha) > 0$.

Proof. We can cover $A \setminus A_N(\alpha)$ by balls $B_x = B(x, r_x)$, $x \in A \setminus A_N(\alpha)$ satisfying

$$\lambda_N(C_0 \cap B_x \setminus A) > (1 - \alpha) \lambda_N(C_0 \cap B_x).$$

By the Besicovich's covering theorem (cf. [Mat95], p. 30) there is an integer ν_N depending only on N such that we can extract a countable sub-family $\{B_{x_j}\}_{j \geq 1}$ of balls that are ν_N to ν_N disjoint and still cover $A \setminus A_N(\alpha)$. It follows that

$$\lambda_N(A \setminus A_N(\alpha)) \leq \sum_j \lambda_N(C_0 \cap B_{x_j}) \leq \sum_j (1 - \alpha)^{-1} \lambda_N(C_0 \cap B_{x_j} \setminus A)$$

and so

$$\lambda_N(A \setminus A_N(\alpha)) \leq \nu_N (1 - \alpha)^{-1} \lambda_N(C_0 \setminus A).$$

For the measurability of $A_N(\alpha)$ it is enough to observe that

$$A_N(\alpha) = \bigcap_{r \in \mathbb{Q}_+^*, \beta \in \mathbb{Q}_+^*, \beta < \alpha} \{x \in A : \lambda_N(A \cap B(x, r) \cap C_0) > \beta \lambda_N(B(x, r) \cap C_0)\}$$

and that every $A_N(\beta, r) := \{x \in A : \lambda_N(A \cap B(x, r) \cap C_0) > \beta \lambda_N(B(x, r) \cap C_0)\}$ is relatively open in A . □

Lemma 4.10. *We keep the notations and assumptions of Lemma 4.9 and fix a decomposition $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^p$. Let $A_N(m, \alpha)$ be the set of points $x = (x', x'')$ in A such that with respect to the slice $C_0 \cap \{x'\} \times \mathbb{R}^p$, $A_{x'} = \{y''; (x', y'') \in A\}$ is of density $\geq \alpha$. Then $A_d(m, \alpha)$ is Borel-measurable and there is a constant $C(m, \alpha)$ such that*

$$\lambda_N(A \setminus A_d(m, \alpha)) \leq C(m, \alpha) \lambda_N(C_0 \setminus A)$$

Proof. The proof of the measurability follows the same lines as above in the proof of Lemma 4.9 and the inequality follows from Lemma 4.9 and Fubini's Theorem. □

REFERENCES

- [AG01] David H. Armitage and Stephen J. Gardiner. *Classical potential theory*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001.
- [AH96] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.
- [Ava61] Vazguen Avanisian. Fonctions plurisousharmoniques et fonctions doublement sousharmoniques. *Ann. Sci. École Norm. Sup. (3)*, 78:101–161, 1961.
- [BH93] Bogdan Bojarski and Piotr Hajłasz. Pointwise inequalities for Sobolev functions and some applications. *Studia Math.*, 106(1):77–92, 1993.
- [Boj91] B. Bojarski. Remarks on some geometric properties of Sobolev mappings. In *Functional analysis & related topics (Sapporo, 1990)*, pages 65–76. World Sci. Publ., River Edge, NJ, 1991.
- [Bre69] Marcel Brelot. *Axiomatique des fonctions harmoniques*, volume 1965 of *Deuxième édition. Séminaire de Mathématiques Supérieures, No. 14 (Été)*. Les Presses de l'Université de Montréal, Montreal, Que., 1969.
- [BT82] Eric Bedford and B. A. Taylor. A new capacity for plurisubharmonic functions. *Acta Mathematica*, 149(1-2):1–40, 1982.
- [Dem] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry*. <http://www-fourier.ujf-grenoble.fr/~demailly/>.
- [DNS10] Tien-Cuong Dinh, Viêt-Anh Nguyên, and Nessim Sibony. Exponential estimates for plurisubharmonic functions and stochastic dynamics. *J. Differential Geom.*, 84(3):465–488, 2010.

- [Hel09] Lester L. Helms. *Potential theory*. Universitext. Springer-Verlag London, Ltd., London, 2009.
- [Hör07] Lars Hörmander. *Notions of convexity*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. Reprint of the 1994 edition.
- [Kis00] Christer O. Kiselman. Ensembles de sous-niveau et images inverses des fonctions plurisousharmoniques. *Bull. Sci. Math.*, 124(1):75–92, 2000.
- [Kli91] Maciej Klimek. *Pluripotential theory*, volume 6 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.
- [Lel98] Pierre Lelong. *Positivity in complex spaces and plurisubharmonic functions/Positivité dans les espaces complexes et fonctions plurisousharmoniques*, volume 112 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University, Kingston, ON, 1998. Edited and with a note by Paulo Ribenboim.
- [Mat95] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
- [Sib99] Nessim Sibony. Dynamique des applications rationnelles de \mathbf{P}^k . In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.
- [Sko72] Henri Skoda. Sous-ensembles analytiques d'ordre fini ou infini dans \mathbf{C}^n . *Bull. Soc. Math. France*, 100:353–408, 1972.

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